MATH 3235 Probability Theory

09/22/22

Jointly distributed r.v.

$X_1 \quad X_2$ are geometric r.v

p

$$\mathbb{P}(X_1 = x_1) = q^{x_1 - 1} p = p_{X_1}(x_1)$$

$$Y = X_1 + X_2$$

$$\mathbb{P}(Y = y) = \sum_{\substack{x_1, x_2 \\ x_1 + x_2 = y}} p(x_1)\, p(x_2) =$$

$$= \sum_{x_1 = 1}^{y-1} p(x_1)\, p(y - x_1) =$$

$$= \sum_{x_1} q^{x_1 - 1} p\, q^{y - x_1 - 1} p =$$

$$= (y-1)\, q^{y-2} p^2$$

$$P_Y(y) = (y-1)\, q^{y-2}\, p^2$$

Negative binomial distribution.

$$Z = \min(X_1, X_2)$$

$$\mathbb{P}(Z < z)$$

$$F(z) = \mathbb{P}(Z \leq z) \quad \text{c.d.f.}$$

$$= \sum_{y \leq z} \mathbb{P}(Z = y)$$

$$P_Z(z) = F(z) - F(z-1)$$

$$\mathbb{P}(Z > z) = 1 - F(z)$$

$$\{Z > z\} = \{\min(X_1, X_2) > z\} =$$

$$\{X_1 > z\} \cap \{X_2 > z\}$$

$$\mathbb{P}(Z > z) = \mathbb{P}(X_1 > z)\, \mathbb{P}(X_2 > z)$$

$$\mathbb{P}(Z > z) = \mathbb{P}(X_1 > z)^2$$

$$\mathbb{P}(X_1 > z) = \sum_{x > z} q^{x-1} p =$$

$$= q^z p \sum_{y=0}^{\infty} q^y = q^z$$

$$F_{X_1}(x_1) = 1 - q^{x_1}$$

$$\mathbb{P}(Z > z) = q^{2z} = (q^2)^z$$

$$X_1 + X_2 = z$$

$$\mathbb{P}(X_1 = x \mid X_1 + X_2 = z) = \frac{1}{z-1}$$

$$x \ngtr z$$

$$X_1 = x_1 \qquad X_2 = x_2 \qquad x_1 + x_1 = z$$

$$\mathbb{P}(X_1 = x_1 \,\&\, X_2 = x_2) = q^{x_1 - 1} p \, q^{x_2 - 1} p =$$

$$= q^{z-2} p^2$$

$\mathbb{P}(X_1 = x_1 \mid X_1 + X_2 = z)$ does not depend on x_1

$\mathbb{P}(\max(X_1, X_2) = \min(X_1, X_2) + t)$

$\approx$ geometric.

Indicator functions.

$(\Omega, \mathcal{F}, \mathbb{P})$

$A \in \mathcal{F}$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Indicator function

Characteristic function

$\mathbb{1}_A$ is a r.v.

Bernoulli r.v.

A^c $\quad \mathbb{1}_{A^c} = \mathbb{1} - \mathbb{1}_A$

$A \cap B$ $\quad \mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B$

$$A \cup B = (A^c \cap B^c)^c =$$

$$= \mathbb{1} - (\mathbb{1} - \mathbb{1}_A)(\mathbb{1} - \mathbb{1}_B) =$$

$$= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B =$$

$$= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}$$

$$\mathbb{P}(A) = \mathbb{E}(\mathbb{1}_A)$$

$$\mathbb{P}(A \cup B) = \mathbb{E}(\mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}) =$$

$$= \mathbb{E}(\mathbb{1}_A) + \mathbb{E}(\mathbb{1}_B) - \mathbb{E}(\mathbb{1}_{A \cap B}) =$$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$a_n \quad n = 0 \ldots \infty$

$$g(s) = \sum_{n=0}^{\infty} a_n s^n$$

$a_n \simeq C^n \qquad \limsup \frac{a^n}{C^n} < +\infty$

$g(s)$ exists for $s < C^{-1}$

Can I recover the a_n from g?

$a_0 = g(0)$

g' exists if g exists.

$$g'(s) = \sum_{n=1}^{\infty} n s^{n-1} a_n$$

$g'(0) = a_1$

$g''(0) = 2a_2$

$$g^{(n)}(0) = n!\, a_n$$

if X is a r.v. such That

$Im(X) \in \mathbb{N}$

Probability generating function of X

$$G_X(s) = \sum_{n=0}^{\infty} p_X(n)\, s^n$$

$$G_X(s) = \mathbb{E}\left(s^X\right).$$

Uniqueness.

$p(n) \to G(s)$

$G(s) \to p(n)$

Theorem X Y

$$G_X(s) = G_Y(s)$$

Then

$$\mathbb{P}(X = k) = \mathbb{P}(Y = k) \quad \forall k.$$

$$p(0) = (1-p) \qquad p(1) = p$$

$$G_X(s) = (1-p) + ps.$$

$$= 1 + p(s-1)$$

$$G_X(0) = (1-p)$$

$$G_X'(s) = p \quad \Rightarrow \quad G_X'(0) = p$$

$$G_X''(s) \equiv 0$$